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Computers and Mathematics with Applications 55 (2008) 2064–2075

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

New configurations of 24 limit cycles in a quintic system

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Received 22 November 2006; received in revised form 31 August 2007; accepted 31 August 2007

Abstract

This paper concerns with the number and distributions of limit cycles in a Z_3-equivariant quintic planar polynomial system. 24 limit cycles are found in this system and two different configurations of them are shown by combining the methods of double homoclinic loops bifurcation, Poincaré bifurcation and qualitative analysis. The two configurations of 24 limit cycles obtained in this paper are new. The results obtained are useful to the study of weakened 16th Hilbert Problem.

Keywords: Double homoclinic loops; Bifurcation; Melnikov function; Stability; Limit cycles

1. Instruction and main results

One of the problem posed by Smale in his "Mathematical problem for the next century" is the Hilbert's 16th problem (see [1]). It is well-known that the first part studies the mutual disposition of maximal number of separate branches of an algebraic curve; the second part studies the questions of the maximal number and relative position of limit cycles of the planar polynomial vector field. In order to obtain more limit cycles and various configuration patterns of their relative dispositions, Li et al. indicated in [1,2] that an efficient method is to perturb the symmetric Hamiltonian systems having maximal number of centers, i.e., to study the weakened Hilbert's 16th problem posed by Arnold [3] in 1977 for the near Hamiltonian system. In [2], Li and Liu obtained 11 limit cycles in a cubic system by using detection function method. Liu, Yang and Jiang [4] gave different perturbations of a cubic system also and found 11 limit cycles with the same distribution as the one found by Li et al. [2]. As to the case of quintic polynomial system, there are some results: In [5], Li et al. found that 24 limit cycles existing in Z_6-equivariant quintic system. In [6–8], 23 limit cycles are found in Z_q-equivariant quintic system where $q = 2, 3, 5, 6$.

As to study of weakened Hilbert's 16th problem, Han et al. [9] first used the idea of changing the stability of homoclinic loop to find limit cycles near a homoclinic loop for quadratic systems. This method was developed to investigate the limit cycles bifurcated from a double homoclinic loops by Han, Chen and Wu [10–13]. A new configuration of 11 limit cycles existing in cubic polynomial system is found by using this method in [11].

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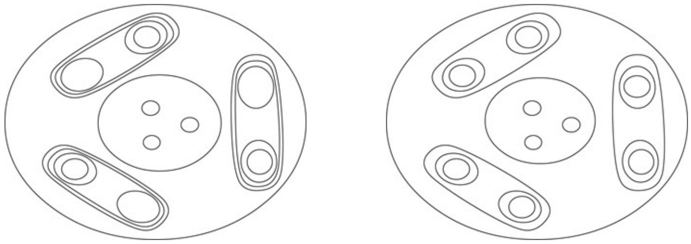
doi:10.1016/j.camwa.2007.08.034



(a) Case 1. (b) Case 2.

Fig. 1. The configurations of 20 limit cycles in system (1).

In this paper, the following near Hamiltonian system is considered

$$\dot{x} = \frac{\partial H(x,y)}{\partial y} + \varepsilon P_5(x,y),$$
$$\dot{y} = -\frac{\partial H(x,y)}{\partial x} + \varepsilon Q_5(x,y), \tag{1}$$

where ε is positive and small,

$$H(x,y) = \frac{x^2}{20} - \frac{x^4}{4} + \frac{3x^6}{20} + \frac{y^2}{20} - \frac{x^2y^2}{2} + \frac{3x^4y^2}{20} - \frac{y^4}{4} + \frac{13x^2y^4}{20} + \frac{7y^6}{60}, \tag{2}$$

the quintic polynomial $P_5(x,y)$, $Q_5(x,y)$ are respectively the real and imaginary part of complex function $F(z,\bar{z})$ given in the following form

$$F(z,\bar{z}) = A_7 z^4 + A_3 \bar{z}^5 + \bar{z}^2(A_1 + A_2 z\bar{z}) + z(A_4 + A_5 z\bar{z} + A_6 z^2\bar{z}^2), \tag{3}$$

where $z = x + iy, \bar{z} = x - iy,\ i^2 = -1,\ A_k = a_k + i\,b_k,\ x, y, a_k, b_k \in \mathbb{R},\ k = 1, 2, \ldots, 7.$

Here we consider the real coefficients $a_i,\ b_i,\ i = 1, 2, \ldots, 7$ as parameters.

From [2,14], we know that the vector field defined by $(P_5(x,y),\ Q_5(x,y))$ is invariant under $2\pi/3$ rotation with respect to the origin O. It is easy to check that system (1) is Z_3-equivariant and from [2], we have the following remark:

Remark 1. As $\varepsilon \neq 0$ system (1) is a Hamiltonian system if and only if

$$a_4 = 0,\ a_5 = 0,\ a_6 = 0,\ a_2 + 4a_7 = 0,\ b_2 - 4b_7 = 0. \tag{4}$$

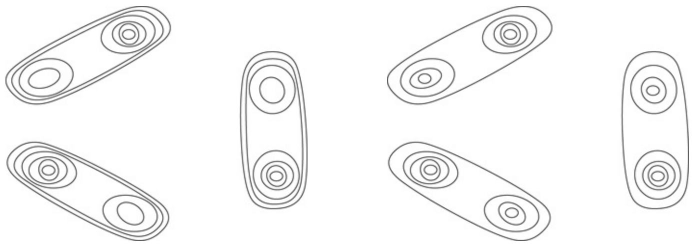
Our main results are stated as follows.

Theorem 1.1. *There exist functions*

$$\begin{aligned}
&\varphi_1(a_4, a_5, b_2, b_7, \varepsilon) \approx -0.2377708a_4 - 0.5504727a_5 + O(\varepsilon),\\
&\varphi_2(a_4, a_5, b_7, \varepsilon) \approx 4.0000000b_7 + O(\varepsilon);\\
&\varphi_3(a_4, a_5, b_7) \approx 1.0655246a_4 + 0.56508131a_5 - 3.0000000b_7;\\
&\phi_1(a_4, a_5, \varepsilon) \approx -0.2866876a_4 - 0.3485818a_5 - 4.0000000a_7 + O(\varepsilon);\\
&\phi_2(a_5, \varepsilon) \approx -0.1926663a_5 + O(\varepsilon)
\end{aligned} \tag{5}$$

such that for fixed $a_5 > 0$, $b_1 > \varphi_3$ and $\varepsilon > 0$ and small, the following two conclusions hold:

(1) *If $0 < a_6 - \varphi_1 \ll b_2 - \varphi_2 \ll \phi_1 - a_2 \ll \phi_2 - a_4 \ll \varepsilon^2$, then system* (1) *has* 20 *limit cycles with their configuration given in* Fig. 1 *(a), where symbol "$\ll$" means much less than.*
(2) *If $0 < a_6 - \varphi_1 \ll \varphi_2 - b_2 \ll \phi_1 - a_2 \ll \phi_2 - a_4 \ll \varepsilon^2$, then system* (1) *has* 20 *limit cycles with their configuration given in* Fig. 1 *(b).*



(a) Case 1. (b) Case 2.

Fig. 2. The configurations of 24 limit cycles in system (1).

Theorem 1.2. *There exist functions φ_1, φ_2, φ_3, ϕ_1 which are given in (5) and*

$$\phi_3(a_5, \varepsilon) \approx -0.9783868 a_5 + O(\varepsilon) \tag{6}$$

such that for fixed $a_5 > 0$, $b_1 > \varphi_3$ and $\varepsilon > 0$ and small, the following two conclusions hold:

(1) *If $0 < a_6 - \varphi_1 \ll b_2 - \varphi_2 \ll \phi_1 - a_2 \ll a_4 - \phi_3 \ll \varepsilon^2$, then system (1) has 24 limit cycles with their configuration given in Fig. 2 (a).*

(2) *If $0 < a_6 - \varphi_1 \ll \varphi_2 - b_2 \ll \phi_1 - a_2 \ll a_4 - \phi_3 \ll \varepsilon^2$, then system (1) has 24 limit cycles with their configuration given in Fig. 2 (b).*

The paper arranges as follows. In Section 2, we briefly describe the properties of the unperturbed system $(1)|_{\varepsilon=0}$ and give its global phase portraits. Lemmas about the existence conditions of double homoclinic loops of system (1) and the quantities determining the stabilities of these double homoclinic loops are also given in this part. The proof of the main results are presented in Section 3.

2. Notations and preliminary lemmas

As $\varepsilon = 0$, system (1) is called unperturbed system which has the form

$$\begin{aligned} \dot{x} &= \frac{y}{10} - x^2 y + \frac{3x^4 y}{10} - y^3 + \frac{13x^2y^2}{5} + \frac{7y^5}{10}, \\ \dot{y} &= -\frac{x}{10} + x^3 - \frac{9x^5}{10} + xy^2 - \frac{3x^3y^2}{5} - \frac{13xy^4}{10}. \end{aligned} \tag{7}$$

In this section, we first describe the phase portraits of the above unperturbed system. By solving polynomial equations, we get that unperturbed system (7) has 25 singular points: centers $O = (0, 0)$, A_i, $i = 1, 2, \dots, 12$ and saddle points S_i, $i = 1, 2, \dots, 12$. The coordinates of these singular points are listed as follows:

$$\begin{aligned}
&A_1(x_{A_1}, y_{A_1}), \quad S_1(1, 0), \quad A_7\left(\frac{1}{3}, 0\right), \quad S_7(x_{S_7}, y_{S_7}),\\
&A_2(0, y_{A_2}), \quad S_2\left(\frac{1}{2}, y_{S_2}\right), \quad A_8\left(\frac{1}{6}, y_{A_8}\right), \quad S_8(0, y_{S_8}),\\
&A_3(-x_{A_1}, y_{A_1}), \quad S_3\left(-\frac{1}{2}, y_{S_2}\right), \quad A_9\left(-\frac{1}{6}, y_{A_8}\right), \quad S_9(-x_{S_7}, y_{S_7}),\\
&A_4(-x_{A_1}, -y_{A_1}), S_4(-1, 0), \quad A_{10}\left(-\frac{1}{3}, 0\right), \quad S_{10}(-x_{S_7}, -y_{S_7}),\\
&A_5(0, -y_{A_2}), \quad S_5\left(-\frac{1}{2}, -y_{S_2}\right), A_{11}\left(-\frac{1}{6}, -y_{A_8}\right), \quad S_{11}(0, -y_{S_8}),\\
&A_6(x_{A_1}, -y_{A_1}), \quad S_6\left(\frac{1}{2}, -y_{S_2}\right), \quad A_{12}\left(\frac{1}{6}, -y_{A_8}\right), \quad S_{12}(x_{S_7}, -y_{S_7}),
\end{aligned}$$

where $x_{A_1} \approx 0.9951296$, $y_{A_1} \approx 0.5745383$, $y_{A_2} \approx 1.1490767$, $y_{S_2} \approx 0.8660254$, $y_{A_8} \approx 0.2886751$, $x_{S_7} \approx 0.2848607$, $y_{S_7} \approx 0.1644644$, $y_{S_8} \approx 0.3289288$.

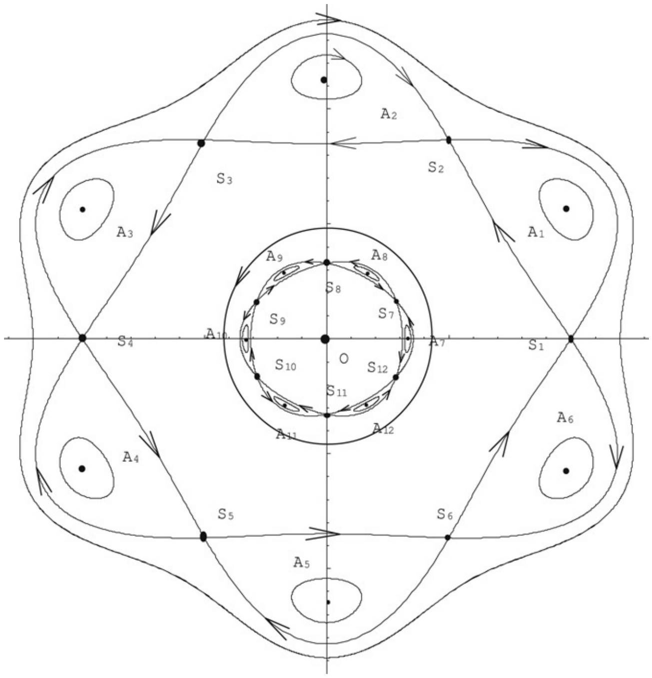


Fig. 3. The phase portraits of unperturbed system (7).

Remark 2. From [2,14], it is easy to check that the unperturbed system (7) is Z_6-equivariant.

The unperturbed system (7) has the first integral of the form $H(x, y) = h$, where function $H(x, y)$ is defined in (2). Then we have $H(A_i) = h_1,\ H(S_i) = h_2, H(O) = 0, H(S_j) = h_3,\ H(A_j) = h_4,\ i = 1, 2, \ldots, 6,\ j = 7, 8, \ldots, 12$, where $h_1 \approx -0.1012704$, $h_2 = -1/20$, $h_3 \approx 0.0026310$, $h_4 = 13/4860$. Especially, the level curves defined by $H(x, y) = h_2$ consist of the six saddle points S_i and six heteroclinic loops denoted by $\Gamma^{h_2}_{i,j} \cup \Gamma^{h_2}_{j,i}$, $j = i + 1$ as $i = 1, 2, \ldots, 5$ and $j = 1$ as $i = 6$, where $\Gamma^{h_2}_{i,j}$ denotes the saddle connection between S_i and S_j with the direction from point S_i to point S_j, $\Gamma^{h_2}_{i,j} \cup \Gamma^{h_2}_{j,i}$ embraces the focus $A_i,\ i < j,\ i = 1, 2, \ldots, 6$. Similarly, the level curves defined by $H(x, y) = h_3$ also consist of the six saddle points S_i and six heteroclinic loops which are denoted by $\Gamma^{h_3}_{i,j} \cup \Gamma^{h_3}_{i,j}$, $j = i + 1$ as $i = 7, 8, \ldots, 11$ and $j = 7$ as $i = 12$, and $\Gamma^{h_3}_{i,j} \cup \Gamma^{h_3}_{i,j}$ embraces the focus $A_i,\ i = 7, 8, \ldots, 12$.

From the above analysis, the phase portraits of unperturbed system (7) are depicted in Fig. 3. (See [15] for more details.)

As $0 < \varepsilon \ll 1$, the number of singular points of unperturbed system (7) is preserved. Denote $A_i(\varepsilon)$, $S_i(\varepsilon)$ the singular points of system (1) near $A_i,\ S_i,\ i = 1, 2, \ldots, 12$ after perturbation. Generally speaking the saddle connections $\Gamma_{i,i+1},\ \Gamma_{i+1,i},\ i, j = 1, 2, \ldots, 12$ of system (7) will break for $0 < \varepsilon \ll 1$. Denote $\Gamma^s_{S_i(\varepsilon)},\ \Gamma^u_{S_i(\varepsilon)}$ the stable and unstable manifold of saddle point $S_i(\varepsilon)$. Recall that the main term of the transversal distance function $d(\Gamma_{i,i+1}, \varepsilon)$ between $\Gamma^u_{S_i(\varepsilon)}$ and $\Gamma^s_{S_{i+1}(\varepsilon)}$ of system (1) is called Melnikov function which is denoted by $M(\Gamma_{i,i+1})$. That is $d(\varepsilon, \Gamma_{i,j}) = \varepsilon N_{i,j} \cdot M(\Gamma_{i,j}) + O(\varepsilon^2)$, where $N_{i,j} > 0$, $0 < \varepsilon \ll 1$.

Remark 3. Noticing the fact that system (1) is Z_3-equivariant, then the following equations hold

$$M(\Gamma_{1,2}) = M(\Gamma_{3,4}) = M(\Gamma_{5,6}), \quad M(\Gamma_{2,1}) = M(\Gamma_{4,3}) = M(\Gamma_{6,5});$$
$$M(\Gamma_{2,3}) = M(\Gamma_{4,5}) = M(\Gamma_{6,1}), \quad M(\Gamma_{3,2}) = M(\Gamma_{5,4}) = M(\Gamma_{1,6});$$
$$M(\Gamma_{7,8}) = M(\Gamma_{9,10}) = M(\Gamma_{11,12}), \quad M(\Gamma_{8,7}) = M(\Gamma_{10,9}) = M(\Gamma_{12,11});$$
$$M(\Gamma_{8,9}) = M(\Gamma_{10,11}) = M(\Gamma_{12,7}), \quad M(\Gamma_{9,8}) = M(\Gamma_{11,10}) = M(\Gamma_{7,12}).$$

From [15], we have the following remark.

Remark 4. The phase portraits of unperturbed system (7) given in Fig. 3 is symmetric with respect to x and y axis.

From Remark 3, we only need to compute the following 8 Melnikov functions $M(\Gamma_{3,2})$, $M(\Gamma_{2,3})$, $M(\Gamma_{6,5})$, $M(\Gamma_{5,6})$, $M(\Gamma_{9,10})$, $M(\Gamma_{10,9})$, $M(\Gamma_{12,7})$ and $M(\Gamma_{7,12})$.

Noticing that system $(1)|_{\varepsilon=0}$ is a Hamiltonian, from [16], as to the expression of the above Melnikov function of the saddle connections, we have the following formulae,

$$M(\Gamma_{i,j}) = \int_{\Gamma_{i,j}} Q_5(x, y(x))\mathrm{d}x - \int_{\Gamma_{i,j}} P_5(x(y), y)\mathrm{d}y. \tag{8}$$

By using Mathematica 4.0, we obtain the following functions $y = y_{32}(x)$, $-x_1 \le x \le 0$, $x = x_{32}(y)$, $y_1 \le y \le y_1^{**}$ and $y = y_{23}(x)$, $-x_1 \le x \le 0$, $x = x_{23}(y)$, $y_1^* \le y \le y_1$ all of which are implicitly determined by the equation $H(x, y) = h_2$, $x_1 = 0.5$, $y_1 = y_{S_2}$, where $y_1^* \approx 0.8482786$, $y_1^{**} \approx 1.3271818$ which are determined by $H(0, y) = h_2$.

Then we get the following results.

Lemma 2.1. *For $0 < \varepsilon \ll 1$, Melnikov functions $M(\Gamma_{3,2})$, $M(\Gamma_{2,3})$, $M(\Gamma_{6,5})$ and $M(\Gamma_{5,6})$ respectively have the following form*

$$\begin{aligned}
M(\Gamma_{3,2}) &\approx 1.4849770a_4 + 2.1700915a_5 + 3.2578737a_6 \\
&\quad - 0.6666667b_1 - 1.2048480b_2 + 2.8193920b_7; \\
M(\Gamma_{2,3}) &\approx -0.8455958a_4 - 0.6898345a_5 - 0.5688086a_6 \\
&\quad + 0.6666667b_1 + 0.5392174b_2 - 0.1568698b_7; \\
M(\Gamma_{6,5}) &\approx 1.4849770a_4 + 2.1700915a_5 + 3.2578737a_6 \\
&\quad + 0.6666667b_1 + 1.2048480b_2 - 2.8193920b_7; \\
M(\Gamma_{5,6}) &\approx -0.8455958a_4 - 0.6898345a_5 - 0.5688086a_6 \\
&\quad - 0.6666667b_1 - 0.5392174b_2 + 0.1568698b_7.
\end{aligned} \tag{9}$$

Proof. From (8) and Remark 4, Melnikov function of the saddle connection $\Gamma_{3,2}$ is computed as follows:

$$\begin{aligned}
\int_{\Gamma_{3,2}} Q_5(x, y)\mathrm{d}x &= \int_{-x_1}^{0} Q_5(x, y_{32}(x))\mathrm{d}x + \int_{0}^{x_1} Q_5(x, y_{32}(-x))\mathrm{d}x \\
&= a_3k_{3,1} + a_4k_{4,1} + a_5k_{5,1} + a_6k_{6,1} + b_1k_{8,1} + b_2k_{9,1} + b_7k_{14,1}; \\
\int_{\Gamma_{3,2}} P_5(x, y)\mathrm{d}y &= \int_{y_1}^{y_1^{**}} P_5(x_{32}(y), y)\mathrm{d}y + \int_{y_1^{**}}^{y_1} P_5(-x_{32}(y), y)\mathrm{d}y \\
&= a_3k_{3,2} + a_4k_{4,2} + a_5k_{5,2} + a_6k_{6,2} + b_1k_{8,2} + b_2k_{9,2} + b_7k_{14,2}.
\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{2,3}$ is computed as follows:

$$\begin{aligned}
\int_{\Gamma_{2,3}} Q_5(x, y)\mathrm{d}x &= \int_{0}^{-x_1} Q_5(x, y_{23}(x))\mathrm{d}x + \int_{x_1}^{0} Q_5(x, y_{23}(-x))\mathrm{d}x \\
&= a_3k_{3,3} + a_4k_{4,3} + a_5k_{5,3} + a_6k_{6,3} + b_1k_{8,3} + b_2k_{9,3} + b_7k_{14,3}; \\
\int_{\Gamma_{2,3}} P_5(x, y)\mathrm{d}y &= \int_{y_1}^{y_1^{*}} P_5(-x_{23}(y), y)\mathrm{d}y + \int_{y_1^{*}}^{y_1} P_5(x_{23}(y), y)\mathrm{d}y \\
&= a_3k_{3,4} + a_4k_{4,4} + a_5k_{5,4} + a_6k_{6,4} + b_1k_{8,4} + b_2k_{9,4} + b_7k_{14,4}.
\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{6,5}$ is computed as follows:

$$\begin{aligned}
\int_{\Gamma_{6,5}} Q_5(x, y)\mathrm{d}x &= \int_{x_1}^{0} Q_5(x, -y_{32}(-x))\mathrm{d}x + \int_{0}^{-x_1} Q_5(x, -y_{32}(-x))\mathrm{d}x \\
&= a_3k_{3,1} + a_4k_{4,1} + a_5k_{5,1} + a_6k_{6,1} - b_1k_{8,1} - b_2k_{9,1} - b_7k_{14,1};
\end{aligned}$$

$$\begin{aligned}\int_{\Gamma_{6,5}} P_5(x,y)\mathrm{d}y &= \int_{-y_1}^{-y_1^{**}} P_5(-x_{32}(-y),y)\mathrm{d}y + \int_{-y_1^{**}}^{-y_1} P_5(x_{32}(-y),y)\mathrm{d}y\\ &= a_3k_{3,2}+a_4k_{4,2}+a_5k_{5,2}+a_6k_{6,2}-b_1k_{8,2}-b_2k_{9,2}-b_7k_{14,2}.\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{5,6}$ is computed as follows:

$$\begin{aligned}\int_{\Gamma_{5,6}} Q_5(x,y)\mathrm{d}x &= \int_{-x_1}^{0} Q_5(x,-y_{23}(x))\mathrm{d}x + \int_{0}^{x_1} Q_5(x,-y_{23}(-x))\mathrm{d}x\\ &= a_3k_{3,3}+a_4k_{4,3}+a_5k_{5,3}+a_6k_{6,3}-b_1k_{8,3}-b_2k_{9,3}-b_7k_{14,3};\\ \int_{\Gamma_{5,6}} P_5(x,y)\mathrm{d}y &= \int_{-y_1}^{-y_1^{*}} P_5(x_{23}(-y),y)\mathrm{d}y + \int_{-y_1^{*}}^{-y_1} P_5(-x_{23}(-y),y)\mathrm{d}y\\ &= a_3k_{3,4}+a_4k_{4,4}+a_5k_{5,4}+a_6k_{6,4}-b_1k_{8,4}-b_2k_{9,4}-b_7k_{14,4}.\end{aligned}$$

By using Mathematica 4.0, we get the following numeric results:

$k_{3,1}\approx -1.5828143,\quad k_{4,1}\approx 1.1755012,\quad k_{5,1}\approx 1.7768947,\quad k_{6,1}\approx 2.7465717,$
$k_{8,1}\approx -1.3173525,\quad k_{9,1}\approx -2.0461061,\quad k_{14,1}\approx 1.5096960;$
$k_{3,2}\approx -1.5828143,\quad k_{4,2}\approx -0.3094758,\quad k_{5,2}\approx -0.3931968,\quad k_{6,2}\approx -0.5113020,$
$k_{8,2}\approx -0.6506858,\quad k_{9,2}\approx -0.8412581,\quad k_{14,2}\approx -1.30969600;$
$k_{3,3}\approx -0.0186579,\quad k_{4,3}\approx -0.8558106,\quad k_{5,3}\approx -0.6986407,\quad k_{6,3}\approx -0.5764552,$
$k_{8,3}\approx 0.6491126, k_{9,3}\approx 0.5240771,\quad k_{14,3}\approx -0.1784349;$
$k_{3,4}\approx -0.0186579,\quad k_{4,4}\approx -0.0102148,\quad k_{5,4}\approx -0.0088063,\quad k_{6,4}\approx -0.0076466,$
$k_{8,4}\approx -0.0175541,\quad k_{9,4}\approx -0.0151403,\quad k_{14,4}\approx -0.0215651.$

From the above numeric results and Eq. (8), we get the Eqs. (9). □

Similarly, we get the functions of the saddle connections $\Gamma_{9,10}$, $\Gamma_{10,9}$: $y = y_{910}(x)$, $-x_2^{**} \le x \le -x_2$, $x = x_{910}(y)$, $0 \le y \le y_2$ and $y = y_{109}(x)$, $-x_2^{*} \le x \le -x_2$, $x = x_{109}(y)$, $0 \le y \le y_2$ which are implicitly determined by the equation $H(x,y) = h_3$, $x_2 = x_{S_7}$, $y_2 = y_{S_7}$, where $x_2^{*} \approx 0.3104400$, $x_2^{**} \approx 0.3549845$ which are determined by $H(x,0) = h_3$.

Then we get the following lemma.

Lemma 2.2. *For $0 < \varepsilon \ll 1$, Melnikov functions $M(\Gamma_{9,10})$, $M(\Gamma_{10,9})$, $M(\Gamma_{12,7})$ and $M(\Gamma_{7,12})$ respectively have the following form*

$$\begin{aligned}M(\Gamma_{9,10}) &\approx 0.0237255a_1+0.0029201a_2-0.1245629a_4\\ &\quad -0.0148497a_5-0.0017742a_6-0.0039795a_7;\\ M(\Gamma_{10,9}) &\approx -0.0237255a_1-0.0023567a_2+0.1051288a_4\\ &\quad +0.0105672a_5+0.0010635a_6-0.0017261a_7;\\ M(\Gamma_{12,7}) &\approx -0.0237255a_1-0.0029201a_2-0.1245629a_4\\ &\quad -0.0148497a_5-0.0017742a_6-0.0039795a_7;\\ M(\Gamma_{7,12}) &\approx 0.0237255a_1+0.0023567a_2+0.1051288a_4\\ &\quad +0.0105672a_5+0.0010635a_6+0.0017261a_7.\end{aligned} \tag{10}$$

Proof. Melnikov function of the saddle connection $\Gamma_{9,10}$ is computed as follows:

$$\begin{aligned}\int_{\Gamma_{9,10}} Q_5(x,y)\mathrm{d}x &= \int_{-x_2}^{-x_2^{**}} Q_5(x,y_{910}(x))\mathrm{d}x + \int_{-x_2^{**}}^{-x_2} Q_5(x,-y_{910}(x))\mathrm{d}x\\ &= a_1k_{1,5}+a_2k_{2,5}+a_3k_{3,5}+a_4k_{4,5}+a_5k_{5,5}+a_6k_{6,5}+a_7k_{7,5};\\ \int_{\Gamma_{9,10}} P_5(x,y)\mathrm{d}y &= \int_{y_2}^{0} P_5(x_{910}(y),y)\mathrm{d}y + \int_{0}^{-y_2} P_5(x_{910}(-y),y)\mathrm{d}y\\ &= a_1k_{1,6}+a_2k_{2,6}+a_3k_{3,6}+a_4k_{4,6}+a_5k_{5,6}+a_6k_{6,6}+a_7k_{7,6}.\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{10,9}$ is computed as follows:

$$\begin{aligned}\int_{\Gamma_{10,9}} Q_5(x,y)\mathrm{d}x &= \int_{-x_2}^{-x_2^*} Q_5(x,-y_{109}(x))\mathrm{d}x + \int_{-x_2^*}^{-x_2} Q_5(x,y_{109}(x))\mathrm{d}x\\ &= a_1k_{1,7}+a_2k_{2,7}+a_3k_{3,7}+a_4k_{4,7}+a_5k_{5,7}+a_6k_{6,7}+a_7k_{7,7};\\ \int_{\Gamma_{10,9}} P_5(x,y)\mathrm{d}y &= \int_{-y_2}^{0} P_5(x_{109}(-y),y)\mathrm{d}y + \int_{0}^{y_2} P_5(x_{109}(y),y)\mathrm{d}y\\ &= a_1k_{1,8}+a_2k_{2,8}+a_3k_{3,8}+a_4k_{4,8}+a_5k_{5,8}+a_6k_{6,8}+a_7k_{7,8}.\end{aligned}$$

From Remark 4, Melnikov function of the saddle connection $\Gamma_{12,7}$ is computed as follows:

$$\begin{aligned}\int_{\Gamma_{12,7}} Q_5(x,y)\mathrm{d}x &= \int_{x_2}^{x_2^{**}} Q_5(x,-y_{910}(-x))\mathrm{d}x + \int_{x_2^{**}}^{x_2} Q_5(x,y_{910}(-x))\mathrm{d}x\\ &= -a_1k_{1,5}-a_2k_{2,5}+a_3k_{3,5}+a_4k_{4,5}+a_5k_{5,5}+a_6k_{6,5}-a_7k_{7,5};\\ \int_{\Gamma_{12,7}} P_5(x,y)\mathrm{d}y &= \int_{-y_2}^{0} P_5(-x_{910}(-y),y)\mathrm{d}y + \int_{0}^{y_2} P_5(-x_{910}(y),y)\mathrm{d}y\\ &= -a_1k_{1,6}-a_2k_{2,6}+a_3k_{3,6}+a_4k_{4,6}+a_5k_{5,6}+a_6k_{6,6}-a_7k_{7,6}.\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{7,12}$ is computed as follows:

$$\begin{aligned}\int_{\Gamma_{7,12}} Q_5(x,y)\mathrm{d}x &= \int_{x_2}^{x_2^{*}} Q_5(x,y_{109}(-x))\mathrm{d}x + \int_{x_2^{*}}^{x_2} Q_5(x,-y_{109}(-x))\mathrm{d}x\\ &= -a_1k_{1,7}-a_2k_{2,7}+a_3k_{3,7}+a_4k_{4,7}+a_5k_{5,7}+a_6k_{6,7}-a_7k_{7,7};\\ \int_{\Gamma_{7,12}} P_5(x,y)\mathrm{d}y &= \int_{y_2}^{0} P_5(-x_{109}(y),y)\mathrm{d}y + \int_{0}^{-y_2} P_5(-x_{109}(-y),y)\mathrm{d}y\\ &= -a_1k_{1,8}-a_2k_{2,8}+a_3k_{3,8}+a_4k_{4,8}+a_5k_{5,8}+a_6k_{6,8}-a_7k_{7,8}.\end{aligned}$$

By using Mathematica 4.0, we get the following numeric results:

$$\begin{aligned}&k_{1,5}\approx -0.0096588,\quad k_{2,5}\approx -0.0011092,\quad k_{3,5}\approx 0.0005259,\quad k_{4,5}\approx -0.0154320,\\ &k_{5,5}\approx -0.0017681,\quad k_{6,5}\approx -0.0002029,\quad k_{7,5}\approx 0.0016047;\\ &k_{1,6}\approx -0.0333843,\quad k_{2,6}\approx -0.0040293,\quad k_{3,6}\approx 0.0005259,\quad k_{4,6}\approx 0.1091309,\\ &k_{5,6}\approx 0.0130816,\quad k_{6,6}\approx 0.0015713,\quad k_{7,6}\approx -0.0023748;\\ &k_{1,7}\approx 0.0033738,\quad k_{2,7}\approx 0.0003498,\quad k_{3,7}\approx -0.0001384,\quad k_{4,7}\approx 0.0057149,\\ &k_{5,7}\approx 0.0005930,\quad k_{6,7}\approx 0.0000616,\quad k_{7,7}\approx -0.0004780;\\ &k_{1,8}\approx 0.0270993,\quad k_{2,8}\approx 0.0027066,\quad k_{3,8}\approx -0.0001384,\quad k_{4,8}\approx -0.0994138,\\ &k_{5,8}\approx -0.0099742,\quad k_{6,8}\approx -0.0010020,\quad k_{7,8}\approx 0.0012481.\end{aligned}$$

From the above numeric results and Eq. (8), we get the Eqs. (10). □

Remark 5. From the definition of the function $d(\varepsilon,\Gamma_{i,j})$, we know that if $d(\varepsilon,\Gamma_{i,j})=0$, $0<\varepsilon\ll 1$, then system (1) has a saddle connection between saddle point $S_i(\varepsilon)$ and $S_j(\varepsilon)$ denoted by $\Gamma_{i,j}(\varepsilon)$ with the orientation from $S_i(\varepsilon)$ to $S_j(\varepsilon)$.

From [17], we have the following lemma.

Lemma 2.3. *As $\varepsilon\neq 0$ small, there exist functions*

$$\begin{aligned}&d(\varepsilon,\Gamma_{3,2},\Gamma_{2,3})=\varepsilon N_1(M(\Gamma_{3,2})+M(\Gamma_{2,3}))+O(\varepsilon^2),\quad N_1>0\\ &d(\varepsilon,\Gamma_{6,5},\Gamma_{5,6})=\varepsilon N_2(M(\Gamma_{6,5})+M(\Gamma_{5,6}))+O(\varepsilon^2),\quad N_2>0\\ &d(\varepsilon,\Gamma_{9,10},\Gamma_{10,9})=\varepsilon N_3(M(\Gamma_{9,10})+M(\Gamma_{10,9}))+O(\varepsilon^2),\quad N_3>0\\ &d(\varepsilon,\Gamma_{12,7},\Gamma_{7,12})=\varepsilon N_4(M(\Gamma_{12,7})+\Gamma(\Gamma_{7,12}))+O(\varepsilon^2),\quad N_4>0\end{aligned}$$

such that

(1) *when* $d(\varepsilon, \Gamma_{3,2}, \Gamma_{2,3}) = 0$, *then system* (1) *has a homoclinic loop denoted by* $\Gamma_{3,2}(S_3(\varepsilon))$ *(resp.,* $\Gamma_{3,2}(S_2(\varepsilon))$*) near* $\Gamma_{3,2} \cup \Gamma_{2,3}$, *which passes through the saddle point* $S_3(\varepsilon)$ *(resp.,* $S_2(\varepsilon)$*) if* $d(\varepsilon, \Gamma_{3,2}) < 0$ *(resp.,* $d(\varepsilon, \Gamma_{3,2}) > 0$*);*
(2) *when* $d(\varepsilon, \Gamma_{6,5}, \Gamma_{5,6}) = 0$, *then system* (1) *has a homoclinic loop denoted* $\Gamma_{6,5}(S_5(\varepsilon))$ *(resp.,* $\Gamma_{6,5}(S_6(\varepsilon))$*) near* $\Gamma_{6,5} \cup \Gamma_{5,6}$, *which passes through the saddle point* $S_5(\varepsilon)$ *(resp.,* $S_6(\varepsilon)$*) if* $d(\varepsilon, \Gamma_{6,5}) > 0$ *(resp.,* $d(\varepsilon, \Gamma_{6,5}) < 0$*);*
(3) *when* $d(\varepsilon, \Gamma_{9,10}, \Gamma_{10,9}) = 0$, *then system* (1) *has a homoclinic loop denoted by* $\Gamma_{9,10}(S_{10}(\varepsilon))$ *(resp.,* $\Gamma_{9,10}(S_9(\varepsilon))$*) near* $\Gamma_{9,10} \cup \Gamma_{10,9}$, *which passes through the saddle point* $S_{10}(\varepsilon)$ *(resp.,* $S_9(\varepsilon)$*) if* $d(\varepsilon, \Gamma_{9,10}) < 0$ *(resp.,* $d(\varepsilon, \Gamma_{9,10}) > 0$*);*
(4) *when* $d(\varepsilon, \Gamma_{12,7}, \Gamma_{7,12}) = 0$, *then system* (1) *has a homoclinic loop denoted by* $\Gamma_{12,7}(S_{12}(\varepsilon))$ *(resp.,* $\Gamma_{12,7}(S_7(\varepsilon))$*) near* $\Gamma_{12,7} \cup \Gamma_{7,12}$, *which passes through the saddle point* $S_{12}(\varepsilon)$ *(resp.,* $S_7(\varepsilon)$*) if* $d(\varepsilon, \Gamma_{12,7}) > 0$ *(resp.,* $d(\varepsilon, \Gamma_{12,7}) < 0$*).*

Let $d(\varepsilon, L_{3,2}, \Gamma_{2,3}) = 0$, $d(\varepsilon, \Gamma_{6,5}, \Gamma_{5,6}) = 0$. Implicit function theorem implies that there exist functions φ_1, φ_2 such that for $\varepsilon > 0$ small,

$$d(\varepsilon, \Gamma_{3,2}, \Gamma_{2,3}) = d(\varepsilon, \Gamma_{6,5}, \Gamma_{5,6}) = 0 \Longleftrightarrow a_6 = \varphi_1,\ b_2 = \varphi_2,$$

where φ_1, φ_2 are given in (5).

When equations $a_6 = \varphi_1$, $b_2 = \varphi_2$ hold, then we have

$$M(\Gamma_{3,2}) \approx 0.7103497a_4 + 0.3767209a_5 - 0.6666667b_1 - 2.0000000b_7,$$
$$M(\Gamma_{6,5}) \approx 0.7103497a_4 + 0.3767209a_5 + 0.6666667b_1 + 2.0000000b_7.$$

Let $M(\Gamma_{3,2}) = 0$, we get $b_1 = \varphi_3(a_4, a_5, b_7)$, where φ_3 is given in (5). Therefore, if $b_1 > \varphi_3$, then $d(\varepsilon, \Gamma_{3,2}) < 0$, $d(\varepsilon, \Gamma_{2,3}) > 0$ for $\varepsilon > 0$ small.

Then from Z_3-equivariance of system (1) and Lemma 2.3, we have

Lemma 2.4. *Suppose* $a_6 = \varphi_1$, $b_2 = \varphi_2$, *and* $b_1 > \varphi_3$. *Then for* $\varepsilon > 0$ *small, system* (1) *has three double homoclinic loops* $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$, $\Gamma_{3,2}(S_3(\varepsilon)) \cup \Gamma_{3,4}(S_3(\varepsilon))$, $\Gamma_{5,4}(S_5(\varepsilon)) \cup \Gamma_{5,6}(S_5(\varepsilon))$, *where* $\Gamma_{i,j}(S_i(\varepsilon))$ *is the homoclinic loop passing through saddle point* $S_i(\varepsilon)$ *and tending to* $\Gamma_{i,j} \cup \Gamma_{j,i}$, *as* $\varepsilon \to 0$, $i = 1, 3, 5$.

Denote A the point in the inner side of the double homoclinic loops (denoted by Γ_{dhomo}) of system (1). It is known that if ω-set of point A is Γ_{dhomo}, then we call Γ_{dhomo} is isolated and inner stable; if α-set of point A is Γ_{dhomo}, then we call Γ_{dhomo} is inner unstable. Similarly, denote B the point in the outer side of the double homoclinic loops Γ_{dhomo} of system (1). If ω-set of point B is Γ_{dhomo}, then we call Γ_{dhomo} is isolated and outer stable; if α-set of point B is Γ_{dhomo}, then we call Γ_{dhomo} is outer unstable.

As to the stability of the double homoclinic loops of system (1), we have the following lemma.

Lemma 2.5. *Suppose that the parameters of system* (1) *satisfy the following conditions* $a_6 = \varphi_1$, $b_2 = \varphi_2$ *and* $b_1 > \varphi_3$. *Then*

(i) *if* $a_2 - \phi_1 < 0 (> 0)$, *then the double homoclinic loops* $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$ *of system* (1) *are inner and outer stable (unstable), where* ϕ_1 *is given in* (5).
(ii) *if* $a_2 = \phi_1$ *and* $\operatorname{div}(S_2(\varepsilon)) < 0 (> 0)$, *then the double homoclinic loops* $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$ *of system* (1) *are inner and outer stable (unstable).*

Proof. Under the conditions of the lemma, by direct computation, we have

$$e\operatorname{div}(S_1(\varepsilon)) \approx \varepsilon(2a_2 + 0.5733751a_4 + 0.6971637a_5 + 8a_7) + O(\varepsilon^2).$$

Let $\operatorname{div}(S_1(\varepsilon)) = 0$, then we get $a_2 = \phi_1$. From the relationship between the stability of the double homoclinic loops and the sign of divergence quantity of the saddle point, we know the first part of the lemma is true.

When $a_4 = \phi_1$, that is $\operatorname{div}(S_1(\varepsilon)) = 0$, the double homoclinic loops $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$ of system (1) is degenerated. From [10,13,16], we know that its stabilities are determined by the sign of the integrals of the following divergence quantities,

$$\sigma_1 = \varepsilon \oint_{L_{1,2}(S_1(\varepsilon))} \left(\frac{\mathrm{d}P_5}{\mathrm{d}x} + \frac{\mathrm{d}Q_5}{\mathrm{d}y}\right)\mathrm{d}t;$$
$$\sigma_2 = \varepsilon \oint_{L_{1,6}(S_1(\varepsilon))} \left(\frac{\mathrm{d}P_5}{\mathrm{d}x} + \frac{\mathrm{d}Q_5}{\mathrm{d}y}\right)\mathrm{d}t.$$

From the Z_3-equivariance of system (1), we know that $\operatorname{div}(S_2(\varepsilon)) = \operatorname{div}(S_6(\varepsilon))$. Just in the same way in [17], we can prove that $\sigma_1,\ \sigma_2$ and $\operatorname{div}(S_2(\varepsilon))$ have the same sign as $\operatorname{div}(S_2(\varepsilon)) \neq 0$.

From [13], we know that the conclusions of the lemma are true. □

3. Proof of the main results

In the following, we suppose that the parameters of system (1) satisfy the following conditions: $a_5 > 0$, $a_6 = \varphi_1,\ b_2 = \varphi_2,\ b_1 > \varphi_3$ and $a_2 = \phi_1$. From Lemma 2.5, we know that for $0 < \varepsilon \ll 1$ system (1) has 3 degenerated double homoclinic loops whose stabilities are determined by the sign of $\operatorname{div}(S_2(\varepsilon))$.

Denote the Melnikov function of close orbit by $M(\Gamma) = \oint_\Gamma Q_5(x, y)\mathrm{d}x - P_5(x, y)\mathrm{d}y$, where Γ is a close orbit of unperturbed system $(1)|_{\varepsilon=0}$. As $0 < \varepsilon \ll 1$, to study the breaking way of the close orbits Γ_l, Γ_m and Γ_s, which are respectively determined by $H(x, y) = 1$, $H(x, y) = 0$ and $H(x, y) = 0.002$, we give the expressions of Melnikov functions of these three close orbits in the following lemma.

Lemma 3.1. *For $\varepsilon > 0$ small, then Melnikov function $M(\Gamma_l)$, $M(\Gamma_m)$ and $M(\Gamma_s)$ respectively has the following expression*

$$\begin{aligned} &M(\Gamma_l) \approx -4.6915083a_4 - 8.3073995a_5;\\ &M(\Gamma_m) \approx -0.4710828a_4 - 0.0950191a_5;\\ &M(\Gamma_s) \approx 0.1137142a_4 + 0.0059978a_5. \end{aligned} \tag{11}$$

Proof. By using Mathematica 4.0, we get the following functions: $y = y_m(x),\ x_3 \le x \le x_3^*,\ x = x_m(y),\ 0 \le y \le y_3$ which are implicitly determined by equation $H(x, y) = 0$; $y = y_L(x),\ -x_4 \le x \le 0,\ x = x_L(y),\ y_4^* \le y \le y_4$ which are implicitly determined by $H(x, y) = 1$, where $x_3 \approx 0.2410436,\ x_3^* \approx 0.4820873,\ y_3 \approx 0.4174998,\ x_4 \approx 1.4925242,\ y_4 \approx -0.8617093,\ y_4^* \approx -1.7234185$.

For system (1) is invariant under $2\pi/3$ rotation. Then

$$\begin{aligned} M(\Gamma_l) &= 3\int_{x_4}^{0} Q_5(x, y_L(-x))\mathrm{d}x + 3\int_0^{-x_4} Q_5(x, y_L(x))\mathrm{d}x\\ &\quad - 3\int_{y_4}^{y_4^*} P_5(-x_L(y), y)\mathrm{d}y - 3\int_{y_4^*}^{y_4} P_5(x_L(y), y)\mathrm{d}y;\\ M(\Gamma_m) &= 3\int_{x_3}^{x_3^*} Q_5(x, -y_m(x))\mathrm{d}x + 3\int_{x_3^*}^{x_3} Q_5(x, y_m(x))\mathrm{d}x\\ &\quad - 3\int_{-y_3}^{0} P_5(x_m(-y), y)\mathrm{d}y - 3\int_0^{y_3} P_5(x_m(y), y)\mathrm{d}y. \end{aligned}$$

Hence, using numeric computation and noticing the assumption in this section, we get the formulae (11). Similarly, we compute the expression of $M(\Gamma_s)$.

The proof is completed. □

Similarly, we compute the Melnikov functions of close orbits $\Gamma_{A_2},\ \Gamma_{A_5},\ \Gamma_{A_7},\ \Gamma_{A_{10}}$ which respectively only surround the single singular point $A_i,\ i = 2, 5, 7, 10$ in the following lemma, where $\Gamma_{A_i},\ i = 2, 5$ are determined by $H(x, y) = -0.07$ and $\Gamma_{A_i},\ i = 7, 10$ are determined by $H(x, y) = 0.00264$.

Lemma 3.2. *For $\varepsilon > 0$ small, the Melnikov function $M(\Gamma_{A_i})$, $i = 2, 5, 7, 10$ respectively has the following expression*

$$\begin{aligned}
&M(\Gamma_{A_2}) \approx -0.0366517a_4 - 0.0358405a_5;\\
&M(\Gamma_{A_5}) \approx -0.0366517a_4 - 0.0358405a_5;\\
&M(\Gamma_{A_7}) \approx -0.0137441a_4 - 0.0026530a_5;\\
&M(\Gamma_{A_{10}}) \approx -0.0140007a_4 - 0.0029630a_5.
\end{aligned} \tag{12}$$

Proof. By using Mathematica 4.0, we get the following functions: $y = y_{A_2,1}(x)$, $y = y_{A_2,2}$, $-x_5 \le x \le x_5$, $x = x_{A_2,1}(y)$, $x = x_{A_2,2}(y)$, $y_5 \le y \le y_5^*$ which are implicitly determined by equation $H(x, y) = -0.07$; $y = y_{A_7,1}(x)$, $x_6 \le x \le x_7$, $y = y_{A_7,2}(x)$, $x_6 \le x \le x_8$, $x = x_{A_1,1}(y)$, $y_6 \le y \le y_6^*$, $x = x_{A_1,2}(y)$, $-y_6 \le y \le y_6^*$ which are implicitly determined by $H(x, y) = 0.00264$, where $x_5 \approx 0.2813576$, $y_5 \approx 0.9376579$, $y_5^* \approx 1.2929996$, $x_6 \approx 0.3060499$, $y_6^* \approx 0.1175690$, $x_7 = 0.3129959$, $x_8 \approx 0.3526849$. Hence, using numeric computation, then we get

$$\begin{aligned}
&M(\Gamma_{A_2}) \approx 0.3080353a_4 + 0.7621583a_5 + 1.4496609a_6 - 0.3985991b_2 + 1.5943962b_7;\\
&M(\Gamma_{A_5}) \approx 0.3080353a_4 + 0.7621583a_5 + 1.4496609a_6 + 0.3985991b_2 - 1.5943962b_7;\\
&M(\Gamma_{A_7}) \approx -0.0004476a_2 - 0.0139946a_4 - 0.0030920a_5 - 0.0005141a_6 - 0.0017904a_7;\\
&M(\Gamma_{A_{10}}) \approx 0.0004476a_2 - 0.0139946a_4 - 0.0030920a_5 - 0.0005141a_6 + 0.0017904a_7.
\end{aligned}$$

Noticing the assumption in this section, we get the formulae (12). □

It is well-known that as $0 < \varepsilon \ll 1$, the type and stability of singular points $O(0, 0)$, $A_i(\varepsilon)$, $i = 1, 2, \ldots, 12$ of system (1) are closely related with the sign of divergence quantity of the points. Denote $\mathrm{div}(P) = (\frac{\partial P_5}{\partial x} + \frac{\partial Q_5}{\partial y})(P)$ and $V_3(P)$ the divergence quantity and the first-order focus quantity of the point P respectively. Then from focus quantity formulae given in [16] and computing, we get the following lemma.

Lemma 3.3. *As the assumptions given in this section hold, then we get the following formula*

$$\begin{aligned}
&\mathrm{div}(O) = 2a_4\varepsilon + O(\varepsilon^2);\\
&\mathrm{div}(A_2(\varepsilon)) \approx -(0.4871722a_4 + 0.4766428a_5)\varepsilon + O(\varepsilon^2);\\
&\mathrm{div}(A_5(\varepsilon)) \approx -(0.4871722a_4 + 0.4766428a_5)\varepsilon + O(\varepsilon^2);\\
&\mathrm{div}(A_7(\varepsilon)) \approx (2.0036235a_4 + 0.4294896a_5)\varepsilon + O(\varepsilon^2);\\
&\mathrm{div}(A_{10}(\varepsilon)) \approx (1.9611512a_4 + 0.3778478a_5)\varepsilon + O(\varepsilon^2);\\
&V_3(A_2(\varepsilon)) \approx 0.3376571a_5\varepsilon + O(\varepsilon^2), \quad \text{when } \mathrm{div}(A_2(\varepsilon)) = 0;\\
&V_3(A_7(\varepsilon)) \approx 2.5215223a_5\varepsilon + O(\varepsilon^2), \quad \text{when } \mathrm{div}(A_7(\varepsilon)) = 0.
\end{aligned} \tag{13}$$

To determine the relative positions of $\Gamma^u_{S_i(\varepsilon)}$ and $\Gamma^s_{S_j(\varepsilon)}$, $i, j = 1, 2, \ldots, 12$, from Lemmas 2.1 and 2.2 we get the following quantities

$$\begin{aligned}
&M(\Gamma_{2,3}) + M(\Gamma_{5,6}) \approx -1.4206995a_4 - 0.7534417a_5;\\
&M(\Gamma_{3,2}) + M(\Gamma_{6,5}) \approx 1.4206995a_4 + 0.7534417a_5;\\
&M(\Gamma_{7,12}) + M(\Gamma_{10,9}) \approx 0.2097518a_4 + 0.0199635a_5;\\
&M(\Gamma_{12,7}) + M(\Gamma_{9,10}) \approx -0.2482821a_4 - 0.0277462a_5;\\
&M(\Gamma_{7,12}) + M(\Gamma_{12,7}) \approx -0.0191037a_4 - 0.00369498a_5;\\
&M(\Gamma_{10,9}) + M(\Gamma_{9,10}) \approx -0.0194267a_4 - 0.00408774a_5.
\end{aligned} \tag{14}$$

Now by using qualitative analysis of differential equation and perturbation skills, we give the proof of our main results.

The proof of Theorem 1.1. Suppose that the assumptions given in this section hold. Let $\mathrm{div}(A_7(\varepsilon)) = 0$, then we get $a_4 = \phi_2(a_5, \varepsilon)$ which is given in (5). From Lemma 3.3, we have $V_3(A_7(\varepsilon)) > 0$, $\mathrm{div}(S_2(\varepsilon)) > 0$. That means

$A_7(\varepsilon)$ is an unstable fine focus and the double homoclinic loops $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$ are both inner and outer unstable. Under such conditions, we get $M(\Gamma_m) \approx -0.0042573a_5 < 0$, $M(\Gamma_{12,7}) + M(\Gamma_{9,10}) \approx 0.0200894a_5 > 0$. By applying Poincaré–Bendixson theorem, we get that system (1) has one limit cycle surrounding $S_i(\varepsilon)$, $A_i(\varepsilon)$, O, $i = 7, 8, \ldots, 12$. From $M(\Gamma_l) < 0$, $M(\Gamma_{3,2}) + M(\Gamma_{6,5}) > 0$, by applying Poincaré–Bendixson theorem again we know that there exists one limit cycle surrounding all the singular points. From the above analysis, we conclude that system (1) has 2 limit cycles under the above given conditions.

In the following, by using the disturbing skill, we prove that system (1) has 18 more limit cycles in system (1). In first step, fix the value of $a_5 > 0$, slightly change the value of a_4 to satisfy that $0 < \phi_2 - a_4 \ll \varepsilon^2$. At the same time, let $a_6 = \varphi_1$, $b_2 = \varphi_2$, $b_1 > \varphi_3$ and $a_2 = \phi_1$. Then 3 double homoclinic loops of system (1) remain existing and $\operatorname{div}(A_7(\varepsilon)) < 0$. So the focus $A_7(\varepsilon)$ has changed from a stable one to the unstable one. Noticing Z_3-equivariance of system (1) and applying Poincaré–Bendixson theorem, we get 3 limit cycles which respectively circles only $A_i(\varepsilon)$, $i = 7, 9, 11$. In second step, fix the value of a_4 and continue to let $a_6 = \varphi_1$, $b_2 = \varphi_2$, $b_1 > \varphi_3$ and slightly change a_2 to satisfy $0 < \phi_1 - a_2 \ll |a_4 - \phi_2|$. Then from Lemma 2.5, we know that the inner and outer stability of $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$ has changed from unstable into stable. By applying Poincaré–Bendixson theorem again, we get three limit cycles near $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$, one surrounding $A_1(\varepsilon)$, $S_1(\varepsilon)$, $A_6(\varepsilon)$, the other two respectively surrounding $A_i(\varepsilon)$, $i = 1, 6$.

Next, keep $a_6 = \varphi_1$, $b_1 > \varphi_3$, and fix the value of a_2, change b_2 slightly to satisfy that $|b_2 - \varphi_2| \ll a_2 - \phi_1$. Then there are two cases(see [10,13] more details):

Case 1. $0 < a_6 - \varphi_1 \ll b_2 - \varphi_2$. Then there are 2 limit cycles, one of them surrounding $A_6(\varepsilon)$, the other one surrounding $A_1(\varepsilon)$, $A_6(\varepsilon)$, $S_1(\varepsilon)$;

Case 2. $0 < a_6 - \varphi_1 \ll \varphi_2 - b_2$. Then there are 2 limit cycles respectively surrounding $A_1(\varepsilon)$, $A_6(\varepsilon)$.

From the continuous dependence of solutions with respect to the parameters of the differential equation, we know that the limit cycles kept as the parameters are slightly varied. Noticing the fact that system (1) is Z_3-equivariant, we get 15 more limit cycles near the double homoclinic loops of system (1) during the above perturbations. Therefore, totally system (1) has at least 20 limit cycles whose configurations are given in Fig. 1.

The proof is completed. □

The proof of Theorem 1.2. Suppose that the assumptions given in this section hold. Let $\operatorname{div}(A_2(\varepsilon)) = 0$, then we get $a_4 = \phi_3(a_5, \varepsilon)$ which is given in (6). From Lemma 3.3, we have that $V_3(A_2(\varepsilon)) > 0$, $\operatorname{div}(S_2(\varepsilon)) > 0$. That means $A_2(\varepsilon)$ is an unstable fine focus and the double homoclinic loops $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{1,6}(S_1(\varepsilon))$ are both inner and outer unstable. Under such conditions, from Lemma 3.2, we get $M(\Gamma_{A_2}) \approx 0.0000190a_5 > 0$, $M(\Gamma_{A_5}) \approx 0.0000190a_5 > 0$. By applying Poincaré–Bendixson theorem, we get that system (1) has two limit cycles respectively surrounding $A_1(\varepsilon)$, $A_6(\varepsilon)$.

From $\operatorname{div}(O) < 0$, $M(\Gamma_s) < 0$ and $M(\Gamma_{7,12}) + M(\Gamma_{10,9}) < 0$, from $M(\Gamma_{12,7}) + M(\Gamma_{9,10}) > 0$, $M(\Gamma_m) > 0$ and $M(\Gamma_{5,6}) + M(\Gamma_{2,3}) > 0$, from $M(\Gamma_l) < 0$, $M(\Gamma_{3,2}) + M(\Gamma_{6,5}) < 0$, from $\operatorname{div}(A_7(\varepsilon)) < 0$, $\operatorname{div}(A_{10}(\varepsilon)) < 0$ and $M(A_7(\varepsilon))$, $M(A_{10}(\varepsilon))$, $M(\Gamma_{12,7}) + M(\Gamma_{7,12})$, $M(\Gamma_{9,10}) + M(\Gamma_{10,9})$ are all positive, we are not certain that system (1) has any more limit cycle. Therefore, from the above analysis and noticing Z_3-equivariance of system (1), we conclude that system (1) has 6 limit cycles under the above given conditions.

Next, by using same disturbing skill and qualitative analysis method as the one given in the proof of Theorem 1.1, we can prove that system (1) has 3 more limit cycles which respectively surround $A_i(\varepsilon)$, $i = 2, 4, 6$ and 15 more limit cycles near the double homoclinic loops of system (1) with two different configurations. Therefore, system has at least 24 limit cycles and configurations of these 24 limit cycles are given in Fig. 2.

The proof is completed. □

Remark 6. Two configurations of 24 limit cycles of quintic planar polynomial system (1) given in this paper are different from the one given in Ref. [5].

Acknowledgements

The authors would like to express their sincere thanks to the referee's valuable evaluations and comments.

First author's research is supported by fund of Youth of Jiangsu University(05JDG011) and National Natural Science Foundation of China(Grant No 10771088). The second author's research is supported by Shanghai Outstanding Discipline Leader Project (06XD14034) and Shanghai Shuguang Genzong Project (04SGG05).

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